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Noise in Single-Frequency Oscillators and Amplifiers

REIDAR L. KUVÅS

***Abstract*—A generalization of previous oscillator noise analyses has been developed to permit reliable noise characterization of active nonlinear devices. Effects due to sideband correlation in the equivalent noise source are included. A rotating wave approximation (RWA) developed by Lax is used in obtaining the amplitude and phase noise spectra. Conditions are given for phase stabilization of free-running oscillators and for minimum phase noise in phase-locked oscillators and amplifiers. Stability criteria, discussion of spurious sidetones, and effects of a noisy synchronizing signal are given. The noise measure is used to obtain alternative expressions for the noise spectra and the carrier-to-noise ratios of locked oscillators and amplifiers. It is shown that the noise power gain of AM fluctuations is usually much lower than the corresponding gain for FM noise. The theory should be useful in optimizing the noise performance of nonlinear RF generators, such as IMPATT, BARITT, and Gunn diode oscillators.**

I. Introduction

THE NOISE FIGURE is a convenient quantity for specifying the noise characteristics of linear amplifiers, since it is uniquely related to the signal-to-noise ratio (SNR). The situation is more complex in oscillators and large-signal amplifiers. The inherent nonlinearities in such components may cause the signal and noise to transform differently. As a result, the SNR can be sensitive to the operating parameters of the nonlinear system in addition to the strengths of the noise sources. Thus noise characterization of oscillators and large-signal amplifiers is a nontrivial problem.

An informative study on noise in free-running and phase-locked oscillators was presented by Kurokawa [1]. This study gives detailed results for the spectra of the amplitude and phase noise (the spectra for the free-running case were originally derived by Edson [2]), for the possible improvement in FM noise by phase locking, and for the adverse effects of a noisy synchronizing signal. In addition, Kurokawa's theory describes the asymmetry of the noise spectrum and an expected increase in the phase noise when the synchronizing frequency in locked oscillators differs from the free-running frequency. This initial study did not include the frequency dependence of the elements in the equivalent circuit and the derivative of the reactance with respect to the level of operation. These restrictions were removed in a subsequent analysis [3].

The rotating wave approximation (RWA) employed by Lax [4] is very effective in permitting a general but simple formulation of the oscillator noise problem. He derived conditions for decoupling amplitude and phase fluctuations that give a minimum phase noise. A simple analytic expression was obtained for the linewidth, even in the case when these conditions were not met. Important results were given for amplitude fluctuations.

Advantage will be taken of the RWA method in our derivations to obtain generalized results for locked oscillators and amplifiers. For completeness, some of the results obtained by Lax for free-running oscillators will be rederived.

Previous noise analyses have not considered the possibility of correlation between the sidebands of the noise sources. In Appendix I it will be shown that mixing effects in nonlinear systems in general will introduce finite correlation factors. In fact, full sideband correlation has been calculated for the specific case of an IMPATT diode in large-signal operation [5]. Therefore, the present analysis has been generalized to include effects of correlated noise sidebands.

II. Circuit Model

The temporal variation of the RF signal in a well-designed single-frequency oscillator or amplifier is close to being a pure sinusoid. Thus the RF circuit current $I(t)$ of instantaneous amplitude $A(t)$ and phase $\phi_1(t)$ can be written:

$$I(t) = A(t) \cos[\omega_0 t + \phi_1(t)] = A_0[1 + a(t)] \cos[\omega_0 t + \phi_0 + \phi(t)] \qquad (1)$$

where ω_0 is the signal frequency and A_0 and ϕ_0 represent the amplitude and the phase in the noise-free case. In the presence of noise, slow variations are introduced into the amplitude and phase, which in (1) are described by the normalized amplitude fluctuation $a(t) = [A(t) - A_0]/A_0$ and the phase fluctuation $\phi(t) = \phi_1(t) - \phi_0$, respectively.

The nonlinear behavior of oscillators and large-signal amplifiers in general introduces some higher harmonic content

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The author is with Bell Telephone Laboratories, Inc., Reading, Pa. 19604.

into the RF current $I(t)$. This fact can be accounted for by allowing small amounts of higher harmonics in the amplitude and phase variations. However, the time integration involved in deriving the noise spectra around the fundamental frequency ω_0 will effectively eliminate any contribution from the higher harmonic signals to these spectra. Thus the higher harmonics can be ignored altogether.

The resulting RF equivalent circuit is shown in Fig. 1. The noise sources have been lumped into a noise voltage generator V_n. The other voltage source V_s represents the input voltage for an amplifier or the locking signal for a locked oscillator, and is zero for a free-running oscillator. The impedance Z_d represents a negative-resistance device and is a function of frequency ω and the level of operation through the amplitude $A(t)$ of the RF current. Finally, Z_l is the load impedance presented by the external circuit, which depends only on the frequency.

The Fourier transforms of the voltage and current are related by the circuit equation

$$V(\omega) = V_s(\omega) + V_n(\omega) = Z(\omega, A)I(\omega) \tag{2}$$

where the total impedance is given by

$$Z(\omega, A) = Z_d(\omega, A) + Z_l(\omega) = Z^*(-\omega, A). \tag{3}$$

The last relationship follows from the reality condition on the voltage and current in the time domain.

It is expedient for the derivation of the noise spectra to transform the circuit equation to the time domain

$$\begin{aligned} V(t) &= \int_{-\infty}^{\infty} d\omega e^{j\omega t} V(\omega) \\ &= \int_{-\infty}^{\infty} d\omega e^{j\omega t} Z(\omega, A) I(\omega) \\ &= \int_{-\infty}^{\infty} d\omega Z[-j(d/dt), A] e^{j\omega t} I(\omega) \\ &= Z[-j(d/dt), A] I(t) \end{aligned} \tag{4}$$

where the impedance operator $Z[-j(d/dt)]$ fulfills

$$Z[-j(d/dt)]e^{j\omega t} = e^{j\omega t} Z(\omega). \tag{5}$$

In particular, we shall need the expansion of the impedance operator around ω_0 and A_0:

$$\begin{aligned} Z[-j(d/dt), A] A(t) e^{j[\omega_0 t + \phi(t)]} &= e^{j\omega_0 t} Z[\omega_0 - j(d/dt), A] A(t) e^{j\phi(t)} \\ &\approx e^{j\omega_0 t} \left\{ Z(\omega_0, A_0) - j \frac{\partial Z(\omega_0, A_0)}{\partial \omega_0} \frac{d}{dt} \right. \\ &\quad \left. + (A - A_0) \frac{\partial Z(\omega_0, A_0)}{\partial A_0} \right\} A(t) e^{j\phi(t)} \end{aligned} \tag{6}$$

where a Taylor expansion to the first order was used to obtain the last equation. It is usually permissible to truncate the expansion at first order, since it will be assumed that the large-signal quantities are much larger than the noise amplitudes, i.e., $|a|$ and $|\phi| \ll 1$. Possible exceptions exist if the impedance locus versus the frequency contains loops. In this case, the higher order frequency derivatives of the impedance can become sufficiently large that higher order terms cannot be neglected, in spite of the slow time variation of the amplitude and the phase. However, such operating points are potentially unstable [3] with possible frequency jumping, parasitic oscillations, and broad-band noise generation, which is inconsistent with the concept of a well-designed single-frequency oscillator or amplifier. We shall limit our consideration to well-behaved impedance functions with a corresponding well-defined operating point. However, it is interesting to note that Kenyon [6] has been able to injection lock an oscillator around an impedance loop by using large locking signals. Kurokawa [7] has given an illuminating theoretical stability discussion for this case.

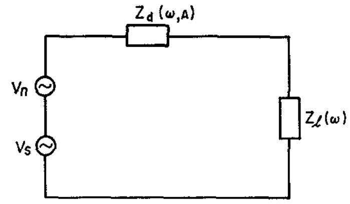

Fig. 1. Equivalent circuit of single-frequency oscillator or amplifier.

It is convenient to introduce the following notation for the expansion coefficients of the linearized impedance operator in (6):

$$Z_0 = Z(\omega_0, A_0) = R_0 + jX_0 = |Z_0| e^{j\alpha} \tag{7}$$

$$Z_A = A_0[\partial Z(\omega_0, A_0)/\partial A_0] = R_A + jX_A \tag{8}$$

$$Z_0 + Z_A = |Z_0 + Z_A| e^{j\theta} \tag{9}$$

$$L = L_1 + jL_2 = -(j/2)[\partial Z(\omega_0, A_0)/\partial \omega_0] \tag{10}$$

where Z_0 is the residual impedance, Z_A gives the amplitude dependence of the nonlinear negative-resistance device, and L is the complex inductance. The reactance corresponding to $2L$ will be written

$$2\omega L = -jZ_\omega = -j(R_\omega + jX_\omega) = |Z_\omega| e^{j\psi} \operatorname{sgn}(\omega) \tag{11}$$

where ω here represents a frequency deviation from ω_0. The function sgn (ω) has the value $+1$ for $\omega > 0$ and -1 for $\omega < 0$.

By using these definitions and the linear expansion (6), we can rewrite (4) as

$$V(t) = (1/2)\{e^{j\omega_0 t} \tilde{Z} A(t) e^{j\phi_1(t)} + e^{-j\omega_0 t} \tilde{Z}^* A(t) e^{-j\phi_1(t)}\} \tag{12}$$

where $\tilde{Z}$ is the following linear impedance operator:

$$\tilde{Z} = 2L(d/dt) + Z_0 + aZ_A. \tag{13}$$

In deriving this result, the impedance has been assumed to be a function of the instantaneous value of the current. Lax showed that this adiabatic approximation gives erroneous results when the power fluctuations of the RF signal are of the same order of magnitude as the power level [4]. However, the approximation should be well satisfied when the noise power is negligible compared to the RF signal level, which is our case of interest.

III. Rotating Wave Approximation

In general, the source voltage V_s can be written

$$V_s(t) = V_0[1 + v_s(t)] \cos [\omega_0 t + \phi_s(t)] \tag{14}$$

where $v_s(t)$ and $\phi_s(t)$ represent the amplitude and phase fluctuations in the source signal. By comparing (1) and (14) it is seen that ϕ_0 gives the phase shift between V_s and the current response in the noise-free case. The linearized circuit equation takes the form

$$e^{j\omega_0 t}[\tilde{Z}Ae^{j\phi_1} - V_0(1 + v_s)e^{j\phi_s}] + e^{-j\omega_0 t}[\tilde{Z}^*Ae^{-j\phi_1} - V_0(1 + v_s)e^{-j\phi_s}] = 2V_n. \quad (15)$$

In the noise-free case ($V_n = 0$), the two terms on the left-hand side must vanish separately for (15) to be fulfilled at all times. The noise introduces a weak coupling between the two terms. Only the time average of this coupling is of importance, since a time integration is involved in solving the differential equation (15). This time average is entirely negligible due to the opposite polarization $\pm\omega_0$ represented by the two exponential factors. Thus it is justified to make an RWA by multiplying (15) with $e^{-j\omega_0 t}$ and neglecting the second term on the left-hand side. The result is

$$[2L(d/dt) + Z_0 + aZ_A]A_0(1 + a)e^{j(\phi_0+\phi)} - V_0(1 + v_s)e^{j\phi_s} = 2V_n e^{-j\omega_0 t}. \quad (16)$$

Lax previously derived an essentially identical equation for a free-running oscillator [4, eq. (3.22)], which corresponds to the special case $V_0 = 0$, $Z_0 = 0$.

Equation (16) can be simplified by linearizing in the noise amplitudes, since these are assumed to be much smaller than the large-signal quantities. As a result, (16) separates into a steady-state large-signal equation and a small-signal equation for the noise amplitudes. The large-signal equation is given by

$$Z_0 A_0 = Z(\omega_0, A_0)A_0 = V_0 e^{-j\phi_0}. \quad (17)$$

Comparison with (7) shows that the angle α can be replaced by $-\phi_0$ for amplifiers and locked oscillators. For free-running oscillators ($V_0 = 0$), the large-signal amplitude is determined by $Z(\omega_0, A_0) = 0$. In this case, the phase of (16) can be fixed by taking $\phi_0 = 0$.

The noise amplitudes are given by the following complex equation:

$$[2L(d/dt) + Z_0 + Z_A]a(t) + j[2L(d/dt) + Z_0]\phi(t) = Z_0[v_s(t) + j\phi_s(t)] + 2[V_n(t)/A_0]e^{-j(\omega_0 t+\phi_0)}. \quad (18)$$

This equation, which was obtained by making an RWA, will be used next to study the amplitude and phase noise spectra.

IV. Spectra of Amplitude and Phase Fluctuations

A. General Spectra

Equation (18) is easily converted into a set of two coupled differential equations by equating the real and imaginary parts separately. It is important to do this separation before applying a Fourier transformation to obtain a set of linear equations for the noise amplitudes. The result is

$$\begin{bmatrix} R_0 + R_A + jX_\omega & -X_0 + jR_\omega \\ X_0 + X_A - jR_\omega & R_0 + jX_\omega \end{bmatrix} \begin{bmatrix} a(\omega) \\ \phi(\omega) \end{bmatrix} = \begin{bmatrix} R_0 & -X_0 \\ X_0 & R_0 \end{bmatrix} \begin{bmatrix} v_s(\omega) \\ \phi_s(\omega) \end{bmatrix} + A_0^{-1} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} V_n(-\omega_0 + \omega)e^{j\phi_0} \\ V_n(\omega_0 + \omega)e^{-j\phi_0} \end{bmatrix}. \quad (19)$$

The Fourier transforms were obtained in the reference frame where $A_0 e^{j\omega_0 t}$ is stationary. Therefore, the frequency ω represents the deviation from the operating frequency ω_0. For shorthand we shall write $V_{n\pm} \equiv V_n(\omega_0 \pm \omega)$ and use the fact that $V(t)$ is real to write $V_n(-\omega_0 + \omega) = V_n^*(\omega_0 - \omega) = V_{n-}^*$. The following solutions are obtained for the Fourier-transformed noise amplitudes by inverting (19):

$$\begin{aligned} a(\omega) = {} & \Delta^{-1}\{[|Z_0|^2 + j\,\mathrm{Im}\,(Z_0^*Z_\omega)]v_s - j\,\mathrm{Re}\,(Z_0^*Z_\omega)\phi_s\} \\ & + (A_0\Delta)^{-1}\{(Z_0 + Z_\omega)V_{n-}^*e^{j\phi_0} \\ & + (Z_0^* - Z_\omega^*)V_{n+}e^{-j\phi_0}\} \end{aligned} \quad (20)$$

$$\begin{aligned} \phi(\omega) = {} & \Delta^{-1}\{[\mathrm{Im}\,(Z_0 Z_A^*) + j\,\mathrm{Re}\,(Z_0^*Z_\omega)]v_s \\ & + [\mathrm{Re}\,[Z_0^*(Z_0 + Z_A)] + j\,\mathrm{Im}\,(Z_0^*Z_\omega)]\phi_s\} \\ & + j(A_0\Delta)^{-1}\{(Z_0 + Z_A + Z_\omega)V_{n-}^*e^{j\phi_0} \\ & - (Z_0^* + Z_A^* - Z_\omega^*)V_{n+}e^{-j\phi_0}\} \end{aligned} \quad (21)$$

where Δ is the system determinant of (19) and is given by

$$\Delta = \mathrm{Re}\,[Z_0^*(Z_0 + Z_A)] - |Z_\omega|^2 - j\,\mathrm{Im}\,[Z_\omega^*(2Z_0 + Z_A)]. \quad (22)$$

From these results we observe that both the voltage and phase noise of the input signal V_s in general couple into the noise spectra of the current response. It is also seen that $a(\omega) = a^*(-\omega)$ and $\phi(\omega) = \phi^*(-\omega)$, which is expected, since $a(t)$ as well as $\phi(t)$ are real time functions. An important implication of these relationships is that the sidebands of both the amplitude and phase fluctuations are fully correlated, i.e., the noise sidebands will add on a voltage rather than a power basis in double-sideband (DSB) detection. Apparently, this observation is valid whenever it is permissible to linearize in the noise amplitudes. Thus full sideband correlation can be expected even for a small-signal amplifier as long as the signal level is much higher than the noise level, which is true for most practical applications.

The noise spectra are found by multiplying (20) and (21) with their respective complex conjugate expressions. The spectra close to the operating frequency are of main interest, i.e.,

$$|V_n(\omega_0)|^2 = |V_n|^2 \approx |V_{n-}|^2 \approx |V_{n+}|^2. \quad (23)$$

Moreover, terms of the form $\langle V_{n-}V_{n+}\rangle$ arise in calculating the spectra. These can be expressed in terms of the complex correlation coefficient for the noise voltage (see Appendix I), which will be written as

$$\rho_v = \rho_1 + j\rho_2 = |\rho_v|\,e^{j\chi} = \frac{\langle V_{n-}V_{n+}\rangle}{[|V_{n-}|^2|V_{n+}|^2]^{1/2}} \approx \frac{\langle V_{n-}V_{n+}\rangle}{|V_n|^2}. \quad (24)$$

This type of correlation arises from nonlinear interactions in the negative-resistance device as shown in Appendix I. It should not be confused with the sideband correlations of the amplitude and phase noise of the circuit current, which were discussed earlier in this section. A third type of correlation coefficient giving the cross correlation between the amplitude and phase fluctuations $\langle a^*\phi\rangle$ is investigated in Appendix II. This correlation is simpler to measure than the other sideband correlations. It can be exploited in experimental noise studies to yield information about the nonlinear device-circuit interaction [8], [9].

The noise of the input voltage will be ignored in giving the general noise spectra to focus attention on the noise response of $V_n(t)$. We shall return later to the effects of a noisy input signal in the discussion of the noise in amplifiers and locked oscillators.

The results for the general noise spectra are

$$|a|^2 = \frac{2|V_n|^2}{A_0^2} \cdot \frac{|Z_0|^2 + |Z_\omega|^2 + \text{Re}\,[\rho_v^* e^{2j\phi_0}(Z_0^2 - Z_\omega^2)]}{\{\text{Re}\,[Z_0^*(Z_0+Z_A)] - |Z_\omega|^2\}^2 + \{\text{Im}\,[Z_\omega^*(2Z_0+Z_A)]\}^2} \tag{25}$$

$$|\phi|^2 = \frac{2|V_n|^2}{A_0^2} \cdot \frac{|Z_0+Z_A|^2 + |Z_\omega|^2 - \text{Re}\,\{\rho_v^* e^{2j\phi_0}[(Z_0+Z_A)^2 - Z_\omega^2]\}}{\{\text{Re}\,[Z_0^*(Z_0+Z_A)] - |Z_\omega|^2\}^2 + \{\text{Im}\,[Z_\omega^*(2Z_0+Z_A)]\}^2}. \tag{26}$$

These expressions reduce to the corresponding results derived by Kurokawa [3, eqs. (36) and (37)] for sufficiently high gains $|Z_0| \ll |Z_A|$ (note that $|Z_0| \to 0$ as the gain approaches infinity) and in the limit of vanishing sideband correlation $\rho_v = 0$ for the noise voltage.

B. *Resonant Case with Real Inductance*

As the expressions (25) and (26) are rather complex, we shall attempt to gain qualitative information about the noise spectra by investigating the special case of resonance $Z_0 = R_0$, assuming a real inductance $L(Z_\omega = jX_\omega)$ and a reactance being independent of the amplitude $Z_A = R_A$. The spectra simplify to

$$|a(\omega)|^2 = \frac{2|V_n|^2}{A_0^2} \cdot \frac{1 + \rho_1[(R_0^2 - X_\omega^2)/(R_0^2 + X_\omega^2)]}{(R_0+R_A)^2 + X_\omega^2} \tag{27}$$

$$|\phi(\omega)|^2 = \frac{2|V_n|^2}{A_0^2} \cdot \frac{1 - \rho_1[((R_0+R_A)^2 - X_\omega^2)/((R_0+R_A)^2 + X_\omega^2)]}{R_0^2 + X_\omega^2}. \tag{28}$$

It is interesting to note the rather dramatic effect on the noise spectra when the real part ρ_1 of the correlation coefficient for the sidebands of the noise voltage approaches either plus or minus one. In the first case $\rho_1 \to 1$, a strong reduction results in the phase noise close to the carrier, while the amplitude noise is increased by approximately a factor of two. The roles are reversed for $\rho_1 \to -1$. Thus the amplitude and phase noises cannot be minimized simultaneously by the correlation coefficient; in most cases there will be a tradeoff such that minimization of one spectrum leads to an increase in the other. This observation is consistent with Convert's calculations for noise in avalanche diodes [10]. The low-frequency asymptote for the amplitude noise spectrum is valid for frequencies $\omega < (R_0+R_A)/L$ and for $\omega < R_0/L$ for the phase noise spectrum. Above these frequencies, the spectra decrease proportionally to ω^{-2}. These features are borne out by the Bode plot for the noise spectra given in Fig. 2. Curves have been plotted for $\rho_1 \lesseqgtr 0$ and demonstrate the effect of the correlation coefficient of the voltage noise source.

C. *Frequency Poles of the Noise Spectra*

The frequency dependence of the spectra is determined by the poles and zeros of $j\omega$ in (25) and (26). The poles are determined by the characteristic frequencies of the system determinant Δ, which according to (22) can be written as

$$\begin{aligned}\Delta &= R_\alpha^2 - [Z_\omega|^2 + j\,\text{sgn}\,(\omega) R_\beta |Z_\omega| \\ &= R_\alpha^2 - \omega^2|2L|^2 + j\omega R_\beta|2L|.\end{aligned} \tag{29}$$

By using (9), (11), and (17), it is found that the resistances R_α and R_β are given by

$$R_\alpha^2 = |Z_0||Z_0+Z_A| \cos(\phi_0 + \theta) \tag{30}$$

$$R_\beta = |Z_0| \cos(\phi_0 + \psi) + |Z_0+Z_A| \cos(\theta - \psi). \tag{31}$$

The characteristic frequencies are specified by $\Delta = 0$, i.e.,

$$j\omega = [-R_\beta \pm (R_\beta^2 - 4R_\alpha^2)^{1/2}]/|4L|. \tag{32}$$

The Fourier transform leading to (19) involves the factor $e^{j\omega t} = e^{j\omega_r t - \omega_i t}$, where the subscripts denote the real and imaginary parts, respectively. Hence, our solutions are stable, provided the imaginary parts of the characteristic frequencies are both positive or zero, which give the stability conditions $R_\alpha^2 \geq 0$ and $R_\beta \geq 0$. These conditions can be written as

$$\cos(\phi_0 + \theta) \geq 0 \tag{33}$$

$$\cos(\phi_0 + \psi) + |1 + Z_A Z_0^{-1}| \cos(\theta - \psi) \geq 0 \tag{34}$$

which reduce to the results obtained by Kurokawa [3, eq. (22)] in the limit of high gains (note that Kurokawa's definition of θ coincides with the present definition when $Z_0 \to 0$). The equality signs in (33) and (34) correspond to conditionally stable cases.

From (29) or (32) it is found that Δ^{-1} has two different poles for $R_\beta^2 > 4R_\alpha^2$, while complex conjugate poles result for $R_\beta^2 < 4R_\alpha^2$. In the latter case, the resonant frequency and the damping factor are given by $\omega_r = R_\alpha/|2L|$ and $\xi = R_\beta/2R_\alpha$, respectively. In the underdamped case $R_\beta^2 < 2R_\alpha^2$, we get resonant effects around $|Z_\omega| = R_\alpha$ such that

$$|\Delta|_{\max}^{-2} = (R_\alpha/R_\beta)^2[1 - (R_\beta/2R_\alpha)^2]^{-1} \tag{35}$$

for

$$\omega = \omega_r[1 - \tfrac{1}{2}(R_\beta/R_\alpha)^2]^{1/2}. \tag{36}$$

In the conditionally stable cases $R_\alpha = 0$ and $R_\beta = 0$, noise poles are found at $\omega = 0$ and $\omega = \omega_r$, respectively. It is seen that $|\Delta|^2$ is rapidly decreasing as we approach the stability limits, i.e., strong enhancement of the noise spectra occur in marginally stable cases.

V. Free-Running Oscillators

A. *General Spectra*

A phase instability is inherent in free-running oscillators, since no energy difference is involved in phase fluctuations due to phase and amplitude variations being in quadrature. This instability results in phase diffusion, i.e., $\langle|\phi|^2\rangle \propto t$. Thus it may seem inconsistent to linearize in ϕ. However, it can be

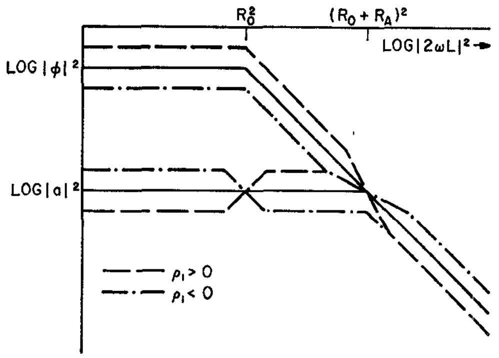

Fig. 2. Bode plot of amplitude and phase noise spectra versus reactance with the real part of the sideband correlation coefficient ρ_v for the voltage noise source as parameter.

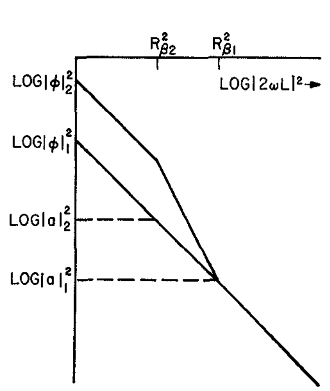

Fig. 3. Bode plot of amplitude and phase noise spectra versus reactance for free-running oscillator. $R_{\beta 1}$ corresponds to cos $(\theta-\psi)=1$ and $R_{\beta 2}$ to cos $(\theta-\psi)<1$.

argued that since the phase diffusion is a slow process, it will modify only the long-term stability and have a negligible effect on the short-term behavior.

Lax avoided a linearization of the phase in his derivation of noise in free-running oscillators [4] because of the phase instability. Our result for the phase noise will turn out to be in agreement with the expression obtained by Lax, which should substantiate the validity of linearizing in the phase as far as the short-term stability is concerned.

For a free-running oscillator ($Z_0=0$), the general expressions for amplitude and phase noise simplify to

$$|a|^2 = \frac{2|V_n|^2}{A_0^2}\cdot\frac{1+|\rho_v|\cos(2\psi-\chi)}{|Z_\omega|^2+|Z_A|^2\cos^2(\theta-\psi)} \tag{37}$$

$$|\phi|^2 = \frac{2|V_n|^2}{|Z_\omega|^2A_0^2}\cdot\frac{|Z_A|^2+|Z_\omega|^2-|\rho_v|[|Z_A|^2\cos(2\theta-\chi)+|Z_\omega|^2\cos(2\psi-\chi)]}{|Z_\omega|^2+|Z_A|^2\cos^2(\theta-\psi)}. \tag{38}$$

Interestingly, there is a pole in the phase spectrum for $\omega\to 0$, which can be ascribed to the phase instability mentioned above. The corresponding pole for the amplitude spectrum is canceled by a zero of the numerator for $\omega=0$. In terms of the discussion in Section IV, the pole is due to $R_\alpha=0$ for $Z_0=0$ corresponding to a conditionally stable case (no restoring force for phase fluctuations).

B. Phase Stabilization

The noise spectra close to the operating frequency $|Z_\omega| \ll R_\beta=|Z_A|$ cos $(\theta-\psi)$ are of particular interest. In this case, the $|Z_\omega|^2$ terms of (37) and (38) can be neglected such that the noise spectra are inversely proportional to $\cos^2(\theta-\psi)$. In this limit, (38) reduces to a result derived by Lax [4, eq. (6.6)] for $\rho_v=0$ by identifying the phase angle of the parameter λ in his theory as $\beta=\theta-\psi$. The noise spectra are minimized for $\theta-\psi=0$ corresponding to Re $[Z_A/L]=0$, which was given as a condition for phase stabilization by Lax.

This criterion can be classified equally well as a general noise stabilization condition for free-running oscillators, as it is seen by dividing (18) by L that the amplitude and phase fluctuations are decoupled in this case (note that $Z_0=0$ for free-running oscillators). The increase in the noise spectra for cos $(\phi-\psi)<1$ is illustrated by the Bode plots in Fig. 3.

C. Effect of the Correlation Coefficient ρ_v

It might appear from (37) and (38) that in contrast to the discussion in Section IV the amplitude and phase noise spectra can be reduced simultaneously for $|Z_\omega|\ll|Z_A|$, when $|\rho_v|\to 1$ in the special case cos $(2\theta-\chi)=-$cos $(2\psi-\chi)=1$. However, this case corresponds to $2\theta=\chi$ and $\psi=\chi+\pi$, which gives $\theta-\psi=\pi/2$ such that $R_\beta=|Z_A|$ cos $(\theta-\psi)\to 0$. The denominators of (37) and (38) decrease at the same rate as the numerators for $\theta-\psi\to\pi/2$ resulting in no net advantage in the noise. It is still possible to achieve separate minimization of either the amplitude or phase noise at the expense of increasing the other spectrum. The chances to exploit these interesting possibilities for reducing either the amplitude or the phase noise spectrum may be remote, since the correlation coefficient and the phase angles θ and ψ cannot be easily manipulated. Thus such noise reduction would in a practical case be dependent on fortuitous circumstances rather than on design.

VI. Phase-Locked Oscillators and Amplifiers

A. Noise Spectra

It is appropriate to consider locked oscillators and amplifiers simultaneously, since they are both characterized by the total circuit impedance being different from zero, i.e., $Z_0\neq 0$. One difference is that a locked oscillator will have a finite locking bandwidth due to its intrinsic instability, while an amplifier is unconditionally stable in the absence of an input signal.

We are mainly interested in the noise close to the operating frequency such that Z_ω can be neglected, since $Z_0\neq 0$. The response of the noise of the input signal shall be included, which means that the expressions (20) and (21) for the noise amplitudes must be used to calculate the spectra. Negligible correlation should exist between the noise of the input signal and the intrinsic noise of the negative-resistance device. Under these conditions, (20) and (21) can be squared to give the following noise spectra for amplifiers and locked oscillators:

$$|a|^2 = \frac{|v_s|^2+(2|V_n|^2/V_0^2)(1+|\rho_v|\cos\chi)}{|1+Z_AZ_0^{-1}|^2\cos^2(\phi_0+\theta)} \tag{39}$$

$$|\phi|^2 = |\phi_s|^2 + \tan^2(\phi_0+\theta)\,|v_s|^2 - 2\tan(\phi_0+\theta)\,\mathrm{Re}\,\langle v_s^*\phi_s\rangle + \frac{2|V_n|^2}{V_0^2}\cdot\frac{1-|\rho_v|\cos(2\phi_0+2\theta-\chi)}{\cos^2(\phi_0+\theta)}. \quad (40)$$

These spectra are minimized for $\phi_0+\theta=0$. Thus the minimum noise is obtained at midband ($X_0=0$) only if the device reactance is independent of the large-signal amplitude ($X_A=0$). The amplitude spectrum arising from the intrinsic noise of the device is, aside from effects of sideband correlation, reduced by the factor $|1+Z_A A_0^{-1}|^{-2}$ compared to the corresponding phase noise.

B. Effects of a Noisy Input Signal

The addition to the amplitude noise spectrum due to a noisy input signal is proportional to $|v_s|^2$. The contribution from the phase noise can be neglected close to the carrier. The full phase noise of the input signal will remain in the amplified signal. In addition, amplitude fluctuations will be partly converted into phase noise unless $\phi_0+\theta=0$, which is identical to the condition for minimum phase noise when the input signal is noise free. The cross correlation of the amplitude and phase fluctuations of the input signal $\langle v_s^*\phi_s\rangle$ in (40) can be determined from (55) in Appendix II.

C. Validity of Spectra

Equations (39) and (40) were obtained by neglecting $|Z_\omega|$ so that the denominator is approximated by $\Delta \approx R_\alpha^2$. In the overdamped case ($R_\beta^2 > 2R_\alpha^2$), this approximation is valid when $|Z_\omega| \ll R_\alpha^2/R_\beta$, while the corresponding condition in the undamped case ($R_\beta^2 < 2R_\alpha^2$) is $|Z_\omega| \ll R_\alpha$. The approximations for the numerators give the additional restriction of $|Z_\omega| \ll |Z_0|$ for the amplitude and $|Z_\omega| \ll |Z_0+Z_A|$ for the phase spectrum to be valid. The first two inequalities determine the frequency range over which the magnitude of the denominator for a phase-locked oscillator is significantly increased above the corresponding value for a free-running oscillator—thus specifying the bandwidth of improved phase noise by injection locking.

D. Spurious Sidetones

The definitions (30) and (31) for R_α and R_β show that a high gain ($Z_0 \to 0$) corresponds to the overdamped case. Underdamping is possible for low-gain amplifiers or injection-locked oscillators with high locking power, depending on the phase angles ϕ_0, θ, and ψ. In Section IV it was pointed out that strong resonant effects occur at $\omega_r = R_\alpha/|2L|$ if $R_\beta \to 0$ [see (35) and (36)]. Thus spurious sidetones can be expected at $\omega_0 \pm \omega_r$ in this limit.

As mentioned in Section II, Kenyon [6] has shown experimentally that phase locking is possible around loops in the impedance locus with high locking powers. Spurious sidetones were observed in such locking. Since large variations of the phase angles are expected in this case, it appears likely that the sidebands are due to $R_\beta \le 0$ in part of the impedance loop.

E. Comparison of Free-Running and Phase-Locked Oscillator

A comparison of the noise spectra of a free-running and phase-locked oscillator is shown in Fig. 4 for the special case

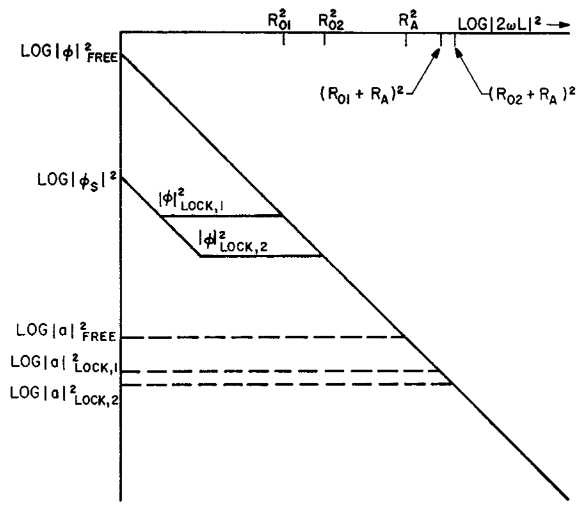

Fig. 4. Comparison of Bode plots of amplitude and phase noise spectra versus reactance for free-running and phase-locked oscillators. R_{01} gives the residual circuit resistance for a higher input power than R_{02}. Note that the phase spectrum for a phase-locked oscillator is determined by the phase noise of the locking source close to the operating frequency.

$\rho_v=0$, $Z_0=R_0$, $Z_A=R_A$, and $Z_\omega=jX_\omega$. No spurious sidebands are expected, since $R_\beta^2=4R_\alpha^2+R_A^2$. Results are shown for two different levels of the locking signal with $P_{\mathrm{lock},1}<P_{\mathrm{lock},2}$. It is assumed that changes in the ratio $|V_n|^2/A_0^2$ can be ignored. The phase noise of the locking signal dominates $|\phi|^2_{\mathrm{lock}}$ close to the operating frequency. A significant improvement in the phase spectrum is achieved for $\omega<R_0/2L$ by phase locking. The amplitude spectrum is reduced by a factor $1+R_0/R_A$ through injection locking.

VII. Noise Measure

Adler and Haus [12] introduced the noise measure

$$M = (F-1)/(1-G^{-1}) \quad (41)$$

as a more meaningful quantity than the noise figure F [11] to characterize the noise of linear amplifiers. The quantity $G=P_o/P_i$ is the gain of the carrier power with P_o and P_i being the output and input powers, respectively. From (39) and (40) it is found that the corresponding noise power gain $G_n=N_o/N_i$ is given by

$$G_n^{\mathrm{AM}} = G/|1+Z_A Z_0^{-1}|^2 \cos^2(\phi_0+\theta) \quad (42)$$

$$G_n^{\mathrm{FM}} = G \quad (43)$$

for AM and FM noise, respectively. Thus the AM noise gain will in most cases be substantially lower than the carrier power gain. This low gain is indicative of a correspondingly small amplification for sidebands of an AM signal, so that the low gain cannot be expected to improve the AM SNR.

An expression relating the input voltage amplitude V_0 to the input power P_i is needed in calculating the noise measure. From the equivalent circuit in Fig. 1 it is found that

$$V_0^2 = 8|R_d|\,P_i/(1-G^{-1}) \quad (44)$$

where $R_d=\mathrm{Re}\,(Z_d)$ gives the equivalent negative resistance of the active device. The noise figures are obtained by using the noise spectra (39) and (40) and assuming an input noise power of kT_0 corresponding to the standard temperature

290K within the bandwidth B. The thermal noise at the input is white such that the noise power is shared equally between AM and FM fluctuations. The corresponding noise measures are found by using (41):

$$M_{\mathrm{AM}} = \frac{|V_n|^2(1 + |\rho_v| \cos \chi)}{4kT_0B|R_d|} \tag{45}$$

$$M_{\mathrm{FM}} = \frac{|V_n|^2[1 - |\rho_v| \cos(2\phi_0 + 2\theta - \chi)]}{4kT_0B|R_d| \cos^2(\phi_0 + \theta)}. \tag{46}$$

These expressions are natural extensions of the noise measure introduced by Haus and Adler [12] to large-signal amplifiers and locked oscillators. The results describe the noise power added by the intrinsic noise of the device and are independent of the power gain. One additional complication as compared to the linear case is introduced by the correlation coefficient ρ_v. However, it can be interpreted to be an integral part of the intrinsic noise properties. In addition, the phase noise can be enhanced by device–circuit interactions through the factor $1/\cos^2(\phi_0+\theta)$.

It is interesting to compare (45) and (46) to the optimum noise measure derived by DeLoach [13] for negative-conductance amplifiers. He found that

$$M_{\mathrm{opt}} = |V_n|^2/4kT_0B|R_d| \tag{47}$$

which is identical to M_{AM} in the limit $\rho_v=0$ and to M_{FM} in the limit $\rho_v=0$ and $\phi_0+\theta=0$.

It has been established that the noise measure represents a meaningful quantity to characterize the noisiness of active devices. Experimentally, this parameter can be determined by measuring the carrier-to-noise ratio (P/N):

$$M = \left[\frac{2P_i}{kT_0B(P_o/N_o)_{\mathrm{SSB}}} - 1\right] \Big/ (1 - G^{-1}) \tag{48}$$

assuming that the input noise is well described by T_0. For DSB detection, the correlation between the sidebands $a(\omega)$, $a(-\omega)$, and $\phi(\omega)$, $\phi(-\omega)$ must be taken into account:

$$(N/P)_{\mathrm{DSB}} = 2(1 + |\rho_{a,\phi}|)(N/P)_{\mathrm{SSB}}. \tag{49}$$

The factor of 2 results from using only the single-sideband (SSB) bandwidth B. It was pointed out in Section IV that $\rho_{a,\phi}=1$ for all practical purposes $(P \gg N)$, since then $a(\omega) = a^*(-\omega)$ and $\phi(\omega)=\phi^*(-\omega)$. Thus the noise sidebands will add on a voltage rather than a power basis such that $(N/P)_{\mathrm{DSB}} = 4(N/P)_{\mathrm{SSB}}$.

VIII. Summary and Conclusions

The amplitude and phase noise spectra of adiabatic single-frequency oscillators and amplifiers have been studied. An RWA was employed in deriving a linearized operator equation for the noise amplitudes. General expressions were obtained for the amplitude and phase noise spectra by Fourier transformation of this operator equation. Special consideration was given to effects due to sideband correlation in the equivalent noise source. The sideband and cross correlation of the amplitude and phase noises were discussed as well. It was pointed out that full sideband correlation is expected for the amplitude and phase noises of well-designed oscillators and amplifiers.

The correlation coefficient ρ_v of sidebands of the noise source had rather dramatic effects on the noise spectra in the limits Re $(\rho_v)=\rho_1 \to \pm 1$. A sharp reduction of the amplitude noise close to the operating frequency results for $\rho_1 \to -1$ with a corresponding increase in the phase noise, while the roles are reversed for $\rho_1 \to 1$.

The system determinant involved in solving for the noise spectra determines the noise poles and the stability conditions. Resonant effects, i.e., enhancement of the noise spectra, occur at $\omega=0$ when $R_\alpha \to 0$ and at $\omega=\omega_r=R_\alpha/|2L|$ for $R_\beta/R_\alpha \to 0$; see (29)–(36). The onset of these resonances define the stability limits.

The case $R_\alpha=0$ is typical of free-running oscillators and gives rise to a phase instability. The corresponding pole for the amplitude spectrum is canceled by a zero of the numerator. The instability results from no energy difference being involved in phase fluctuations and causes phase diffusion. It was established that the phase noise of a free-running oscillator is minimized for Re $[Z_A/Z_\omega]=0$, which was characterized as "phase stabilization" by Lax [4]. Thus the oscillator is phase stable if simultaneously the circuit resistance is independent of frequency and the reactance is independent of the operating level. Otherwise, the phase noise will be enhanced by the factor $1/\cos^2(\theta-\psi) = |Z_A/Z_\omega|^2/[\mathrm{Im}\,(Z_A/Z_\omega)]^2$ in the vicinity of the operating frequency.

Phase-locked oscillators and amplifiers are both characterized by the residual circuit impedance. Detailed expressions were derived for the noise spectra including the effects of a noisy input signal. The magnitude of the noise spectra was minimized when the residual total impedance and the amplitude sensitivity of the impedance combine to give $\phi_0+\theta=0$. This condition is fulfilled at midband $(X_0=0)$ when the reactance is independent of the large-signal current amplitude.

The addition to the amplitude noise spectrum caused by a noisy input signal is proportional to the amplitude noise of this signal. This amplitude noise will partly convert into phase fluctuations for $\phi_0+\theta \neq 0$. The full phase noise of the locking signal will remain in the amplified signal.

Spurious sidetones can be expected at $\omega_0 \pm \omega_r$ for locked oscillators and amplifiers when $R_\beta \to 0$. This type of instability might well explain experimental findings by Kenyon [6] that sidetones are generated when locking around loops in the impedance locus.

In characterization of device noise it is convenient to quantify the spectra in terms of the noise figure or noise measure. The latter quantity gives a better representation of the intrinsic noise of the active device. The noise power gains of AM and FM fluctuations are in general different. In most cases, the AM noise gain is considerably lower. However, both the noise measures reduce to the optimum result derived by DeLoach [13] in the limit $\rho_v=0$ and $\phi_0+\theta=0$. For $\phi_0+\theta \neq 0$, the FM noise measure is enhanced by the factor $1/\cos^2(\phi_0+\theta)$. Finally, it was pointed out that the noise measure can be determined by experimentally measuring the carrier-to-noise ratio.

The generality of the analysis enabled us to reproduce some of Lax's results for free-running oscillators [4] and to generalize Kurokawa's results for high-gain locked oscillators [3] to take into account finite gains and correlation effects. Our results are valid for any value of the locking gain, and can also be used for large-signal amplifiers. It is concluded that the theory should be useful in characterizing and optimizing the noise performance of active nonlinear devices (e.g., IMPATT, BARITT, and Gunn diodes).

Appendix I

Correlation Coefficient for Sidebands of Noise Voltage Source

The primary noise $F(t)$ is assumed to be random and have a white noise spectrum. In a nonlinear system, the resulting open-circuit noise voltage will be related to the primary noise through a Fourier expansion:

$$V_n(\omega_0 + \omega) = \sum_{m=-\infty}^{\infty} C_{1m} F(m\omega_0 + \omega) \tag{50}$$

where C_{1m} are the coupling coefficients resulting from the nonlinearities. The sideband correlation is determined by the product

$$\begin{aligned} V_n(\omega_0 - \omega) V_n(\omega_0 + \omega) &= \sum_m \sum_k C_{1m} C_{1k} F(m\omega_0 - \omega) F(k\omega_0 + \omega) \\ &= \sum_m \sum_k C_{1,-m} C_{1k} F^*(m\omega_0 + \omega) F(k\omega_0 + \omega). \end{aligned} \tag{51}$$

By using the fact that $F(t)$ has a white noise spectrum and taking the statistical averages in (51), we find

$$\langle V_{n-} V_{n+} \rangle = |F|^2 \sum_{m=-\infty}^{\infty} C_{1,-m} C_{1m} \tag{52}$$

where $V_{n\pm} \equiv V_n(\omega_0 \pm \omega)$. The autocorrelation function at ω_0 is given by

$$|V_n(\omega_0)|^2 = |F|^2 \sum_{m=-\infty}^{\infty} C_{1m} C_{1m}^*. \tag{53}$$

Now the correlation coefficient for the sidebands of the noise voltage source can be found from (52) and (53):

$$\rho_v = \frac{\langle V_{n-} V_{n+} \rangle}{|V_n(\omega_0)|^2} = \frac{\sum_{m=0}^{\infty} (2 - \delta_{0m}) C_{1m} C_{1,-m}}{\sum_{p=0}^{\infty} (2 - \delta_{0p}) |C_{1p}|^2} \tag{54}$$

where we have used the condition $V(\omega_0+\omega) = V^*(-\omega_0-\omega)$. Thus although the primary noise is white, the equivalent voltage noise source of a nonlinear device will in general have finite sideband correlations.

Appendix II

Cross-Correlation Coefficient for Amplitude and Phase Fluctuations

Experimentally, the cross correlation between the amplitude and phase noise is relatively simple to measure. Thus this quantity is important in experimental noise studies [8], [9]. A theoretical expression can be derived from (20) and (21). By ignoring the noise of the input signal, the following expression is found for the cross spectrum:

$$\begin{aligned} \langle a^*\phi \rangle = \frac{2|V_n|^2}{A_0^2 |\Delta|^2} [\mathrm{Im} \{ Z_0 Z_A^* - \rho_v^* e^{2j\phi_0} [Z_0(Z_0 + Z_A) - Z_\omega^2] \} \\ + j \,\mathrm{Re}\, [Z_\omega^*(2Z_0 + Z_A) - \rho_v^* e^{2j\phi_0} Z_\omega Z_A]]. \end{aligned} \tag{55}$$

The cross-correlation coefficient is defined by

$$\rho_{a\phi} = \langle a^*\phi \rangle / [|a|^2 |\phi|^2]^{1/2} \tag{56}$$

and thus, it is fully determined by (55), (25), and (26). Equation (55) shows that the cross-correlation coefficient in general will be complex. The imaginary part can be measured experimentally by introducing a relative phase shift of 90° between the amplitude and phase noise components, as was pointed out by Hashiguchi and Okoshi [9].

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